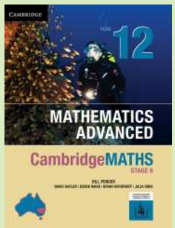
NORMANHURST BOYS HIGH SCHOOL

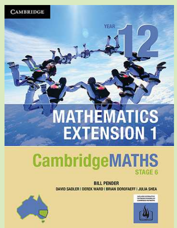
MATHEMATICS ADVANCED
(INCORPORATING MATHEMATICS EXTENSION 1/2)
YEAR 12 COURSE

Topic summary and exercises:

(A) (X1) (X2) **Integration**

With references to





Name: ....

Initial version by R. Trenwith, 1995–2010
Updated by H. Lam, 2011, 2012, with major revision 2013 and subsequently February 2020 for Mathematics Advanced, as well as additional examples from S. Park. Last updated February 23, 2025.
Various corrections by students and members of the Department of Mathematics at North Sydney Boys High School and Normanhurst Boys High School.

Acknowledgements Pictograms in this document are a derivative of the work originally by Freepik at http://www.flaticon.com, used under CC BY 2.0.

Symbols used

- ⚠ Beware! Heed warning.
- (2) Mathematics content.
- (X1) Mathematics Extension 1 content.
- Literacy: note new word/phrase.
- $\mathbb{R}$ the set of real numbers
- $\forall$ for all

Syllabus outcomes addressed

MA12-7 applies the concepts and techniques of indefinite and definite integrals in the solution of problems

Syllabus subtopics

MA-C4 Integral Calculus

Gentle reminder

- For a thorough understanding of the topic, *every* question in this handout is to be completed!
- Additional questions from *CambridgeMATHS Year 12 Advanced* or *CambridgeMATHS Year 12 Extension 1* will be completed at the discretion of your teacher.
- Remember to copy the question into your exercise book!

Learning intentions & outcomes

The anti-derivative (commencing with polynomial-like functions)

✓ **Content/learning intentions**

☐ 21.1 Define anti-differentiation as the reverse of differentiation and use the notation $\int f(x)\,dx$ for anti-derivatives or indefinite integrals (ACMMM114) (ACMMM115)

☐ 21.2 Recognise that any two anti-derivatives of $f(x)$ differ by a constant.

☐ 21.3 Establish and use the formula $\int x^n\,dx = \frac{1}{n+1}x^{n+1} + c$ for $n \neq -1$ (ACMMM116)

☐ 21.4 Establish and use the formula $\int f'(x)\Big(f(x)\Big)^n dx = \frac{1}{n+1}\Big(f(x)\Big)^{n+1} + c$ where $n \neq -1$ (the 'reverse chain rule')

☐ 21.5 (B) Recognise and use linearity of anti-differentiation (ACMMM119)

- Examine families of anti-derivatives of a given function graphically

☐ 21.6 (B) Determine indefinite integrals of the form $\int f(ax+b)dx$ (ACMMM120)

- For this topic, algebraic functions only.

✓ 🎓 **Content/learning intentions**

☐ 21.7 Ⓑ Determine $f(x)$, given $f'(x)$ and an initial condition $f(a) = b$ in a range of practical and abstract applications including coordinate geometry, business and science.

Areas and the definite integral

✓ 🎓 **Content/learning intentions**

☐ 21.8 Know that *the area under a curve* refers to the area between a function and the x axis, bounded by two values of the independent variable and interpret the area under a curve in a variety of contexts

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☐ 21.9 💻 Determine the approximate area under a curve using a variety of shapes including squares, rectangles (inner and outer rectangles), triangles or trapezia

- Consider functions which cannot be integrated in the scope of this syllabus, for example $f(x) = \ln x$, and explore the effect of increasing the number of shapes used.

☐ 21.10 Use the notation of the definite integral $\int_a^b f(x)\,dx$ for the area under the curve $y = f(x)$ from $x = a$ to $x = b$ if $f(x) \geq 0$

☐ 21.11 Understand the relationship of position to signed areas, namely that the signed area above the horizontal axis is positive and the signed area below the horizontal axis is negative

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✓ Content/learning intentions

☐ 21.12 Using technology or otherwise, investigate the link between the anti-derivative and the area under a curve

- Interpret $\int_a^b f(x)\,dx$ as a sum of signed areas (ACMMM127)
- Understand the concept of the signed area function $F(x) = \int_a^x f(t)\,dt$ (ACMMM129)

☐ 21.13 (B) Use the formula $\int_a^b f(x)dx = F(b) - F(a)$, where $F(x)$ is the anti-derivative of $f(x)$, to calculate definite integrals (ACMMM131)

- Understand and use the Fundamental Theorem of Calculus, $F'(x) = \frac{d}{dx}\left(\int_a^x f(t)\,dt\right) = f(x)$ and illustrate its proof geometrically (ACMMM130)
- Use symmetry properties of even and odd functions to simplify calculations of area
- Recognise and use the additivity and linearity of definite integrals (ACMMM128)

☐ 21.14 Use geometric ideas to find the definite integral $\int_a^b f(x)\,dx$ where $f(x)$ is positive throughout an interval $a \leq x \leq b$ and the shape of $f(x)$ allows such calculations, for example when $f(x)$ is a straight line in the interval or $f(x)$ is a semicircle in the interval.

✓ Content/learning intentions

- [ ] 21.15 (B) Calculate the area under a curve (ACMMM132)

- [ ] 21.16 Calculate areas between curves determined by any functions within the scope of this syllabus (ACMMM134)

- [ ] 21.17 (B) Use the Trapezoidal rule to estimate areas under curves
  - Use geometric arguments (rather than substitution into a given formula) to approximate a definite integral of the form $\int_a^b f(x)\,dx$, where $f(x) \geq 0$, on the interval $a \leq x \leq b$, by dividing the area into a given number of trapezia with equal widths
  - Demonstrate understanding of the formula

$$\int_a^b f(x)\,dx \approx \frac{b-a}{2n}\Big[f(a) + f(b) + 2\left(f\left(x_1\right) + \cdots + f\left(x_{n-1}\right)\right)\Big]$$

    where $a = x_0$ and $b = x_n$, and the values x_0, x_1, x_2, $\ldots$, x_n are found by dividing the interval $a \leq x \leq$ into n equal sub-intervals

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Rates of change

✓ Content/learning intentions

- [ ] 21.18 Integrate functions and find indefinite or definite integrals and apply this technique to solving practical problems
  - Calculate total change by integrating instantaneous rate of change

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Contents

Section 1

Primitive functions

Learning Goal(s)

Knowledge
Primitive functions

Skills
Find primitives of functions

Understanding
Family of primitives is generated and differ by a constant

By the end of this section am I able to:

21.1 Define anti-differentiation as the reverse of differentiation and use the notation $\int f(x)\,dx$ for anti-derivatives or indefinite integrals

21.2 Recognise that any two anti-derivatives of $f(x)$ differ by a constant.

21.3 Establish and use the formula $\int x^n\,dx = \frac{1}{n+1}x^{n+1} + c$ for $n \neq -1$

21.5 Recognise and use linearity of anti-differentiation

1.1 Obtaining primitive functions

Fill in the spaces

- A *primitive* of a function $f(x)$ is what needs to be .... to obtain $f(x)$.
- If $f(x) = ax^n$, then $f'(x) =$ ....
- Reversing the process, if $f(x) =$ ...., then $F(x) = ax^n$.
- For x^n terms, ... the power by one, and divide by ....
- Always use ....

Example 1

What is a primitive of $y = 3x^2 - 4x + 4$? **Answer:** $x^3 - 2x^2 + 4x$

Theorem 1

If $y = f(x)$ has a primitive function $F(x)$, then in fact there are .... many primitives, all differing by an .... C.

1.2 Second Fundamental Theorem of Calculus

Theorem 2

(Second) Fundamental Theorem of Calculus The primitive function of $f(x)$, can be written as $F(x)$, and related by

$$\frac{d}{dx}(F(x)) = f(x) \quad \leftrightarrow \quad F(x) = \int f(x)\,dx$$

Some properties need to be known:

1.2.1 Preservation under addition/subtraction

Theorem 3

Primitive of sum is the sum of primitives:

$$\int (f(x) \pm g(x))\ dx = \int f(x)\,dx \pm \int g(x)\,dx$$

Example 2

Find the primitive of $f(x) = 5x^3 + 3x^2 - 4x + 1$. **Answer:** $\frac{5}{4}x^4 + x^3 - 2x^2 + x + C$

1.2.2 Preservation under scalar multiplication

Theorem 4

Move constants out of integral and evaluate.

$$\int af(x)\,dx = a\int f(x)\,dx \qquad (a \in \mathbb{R})$$

Example 3

Find the primitive of $8x^3$. **Answer:** $2x^4 + C$

1.2.3 Constant terms

Theorem 5

Finding the primitive of a constant: look for "invisible" 1s.

$$\int dx = \int 1\,dx = x + C$$

1.3 Further methods of finding primitives

1.3.1 Expansion

Example 4

Find $\displaystyle\int (2x+1)^2\,dx$.

- Expand $(2x+1)^2$: ....
- Find the primitive of each individual term:

  * $\displaystyle\int 4x^2\,dx =$ ....
  * $\displaystyle\int 4x\,dx =$ ....
  * $\displaystyle\int 1\,dx =$ ....

- Sum of primitive terms: ....

Example 5

Find the primitive of $(x-1)(x-2)$. **Answer:** $\frac{1}{3}x^3 - \frac{3}{2}x^2 + 2x + C$

Example 6

Find the primitive of $x^2(1-3x)$ **Answer:** $\frac{1}{3}x^3 - \frac{3}{4}x^4 + C$

Example 7

Evaluate $\displaystyle\int x\sqrt[3]{x}\,dx$. **Answer:** $\frac{3}{7}x^{\frac{7}{3}} + C$

1.3.2 Indices

Important note

Always write in ....

Example 8

Find the primitive of $\sqrt{x} + \sqrt[3]{x}$.

Answer: $\frac{2}{3}x^{\frac{3}{2}} + \frac{3}{4}x^{\frac{4}{3}} + C$

Example 9

Find the primitive of $3 + \dfrac{1}{x^2} - \dfrac{2}{x^3}$.

Answer: $3x - \frac{1}{x} + \frac{1}{x^2} + C$

1.3.3 Splitting fractions (Decomposing rational functions)

Example 10

Evaluate $\displaystyle\int \frac{x^2+2}{x^2}\,dx.$

Answer: $x - 2x^{-1} + C$

Example 11

Evaluate $\displaystyle\int \frac{3x^3-2x^2+x^{-1}}{x^2}\,dx.$

Answer: $\frac{3}{2}x^2 - 2x - \frac{1}{2}x^{-2} + C$

Further exercises

Ⓐ **Ex 3H**

- Q1-7

Ex 4E

- Q1-9, 14

(x1) **Ex 4J**

- Q1-7

Ex 5E

- Q1-9, 14

Further exercises (Legacy Textbooks)

Ex 10J Q1 last 2 columns, Q2-4

Ex 11D Q1, 2(g) - (f), Q3, Q4 LC,

1.3.4 Powers of linear terms

Learning Goal(s)

Knowledge
Indefinite integrals

Skills
Find indefinite integrals

Understanding
Using tangents and normals, find the original function given a point

By the end of this section am I able to:

21.5 Determine indefinite integrals of the form $\int f(ax+b)\,dx$

21.6 Determine $f(x)$, given $f'(x)$ and an initial condition $f(a)=b$ in a range of practical and abstract applications including coordinate geometry, business and science.

Fill in the spaces

- $\dfrac{d}{dx}(ax+b)^n =$ ....
- Hence evaluate $\displaystyle\int (ax+b)^n\,dx$: ....

Example 12

Evaluate $\displaystyle\int (5x+1)^3\,dx$. **Answer:** $\frac{(5x+1)^4}{20}+C$

Example 13

Find the primitive to $\dfrac{1}{(3+6x)^2}$. **Answer:** $-\frac{1}{6}(3+6x)^{-1}+C$

Example 14

[1996 2U HSC Q5] The graph of $y = f(x)$ passes through the point $(1, 3)$ and $f'(x) = 3x^2 - 2$. Find $f(x)$.

Example 15

⚠ Evaluate $\displaystyle\int \frac{2x+1}{(2x-1)^3}\,dx$.

Answer: $-\frac{1}{2}\left(\frac{1}{2x-1} + \frac{1}{(2x-1)^2}\right) + C$

Further exercises

Ⓐ **Ex 3H**

- Q9-11

Ex 4E

- Q11-13, 17-18

(x1) **Ex 4J**

- Q9-11

Ex 5E

- Q11-13, 17-18

Further exercises (Legacy Textbooks)

Ex 10J Q8-9

Ex 11D Q5-6

1.4 Harder primitives: undoing the chain rule

Learning Goal(s)

Knowledge
Chain rule

Skills
Identifying the chain rule residue

Understanding
Why the residue matters in finding primitives

By the end of this section am I able to:

21.4 Establish and use the formula $\int f'(x)\left(f(x)\right)^n dx = \frac{1}{n+1}\left(f(x)\right)^{n+1} + c$ where $n \neq -1$ (the 'reverse chain rule')

1.4.1 Structured questions

Important note

Keyword: Hence. Always use .... when provided.

Example 16

(a) Differentiate $(x^2+1)^4$

(b) Hence find the primitive of $8x\,(x^2+1)^3$.

Example 17

(a) Differentiate $(x^2-5)^6$.

(b) Hence find the primitive of $x\,(x^2-5)^5$.

Example 18

(a) Differentiate $(x^3+5)^4$ with the full setting out of the chain rule.

(b) Hence find the primitive of $12x^2(x^3+5)^3$.

(c) Hence find the primitive of $x^2(x^3+5)^3$.

Example 19

(a) Evaluate $\dfrac{d}{dx}(5-3x)^{\frac{4}{3}}$.

(b) Hence or otherwise, evaluate $\displaystyle\int \sqrt[3]{5-3x}\,dx$

Answer: (a) $-4\sqrt[3]{5-3x}$ (b) $-\frac{1}{4}(5-3x)^{\frac{4}{3}}+C$

1.4.2 Unstructured questions

- Look for derivatives of "inner functions" from the .... inside integrands.

Definition 1

The derivative of "inner functions" appearing due to the .... is also known as the ....

Example 20

Find: $\displaystyle\int x^2\left(x^3+1\right)^4\,dx.$

Answer: $\frac{1}{15}\left(x^3+1\right)^5+C$

Example 21

Find: $\displaystyle\int x\sqrt{1-4x^2}\,dx.$

Answer: $-\frac{1}{12}\left(1-4x^2\right)^{\frac{3}{2}}+C$

Example 22

Find: $\displaystyle\int \frac{x}{\left(5x^2-1\right)^3}\,dx.$

Answer: $-\frac{1}{20}\left(5x^2-1\right)^{-2}+C$

Example 23

Find: $\displaystyle\int \frac{\left(\sqrt{x}+b\right)^2}{\sqrt{x}}\,dx.$

Answer: $\frac{2}{3}\left(\left(\sqrt{x}+b\right)^3\right)+C$

Further exercises

(A) **Ex 4I**

- Q1-14

(X1) **Ex 5I**

- Q1-14
- ⚠ Q16

Further exercises (Legacy Textbooks)

Ex 11H Q1-7

1.4.3 Supplementary exercises

1. Find the primitives of the following functions.

(a) x^5 (b) $3x^8$ (c) $5x$ (d) 7 (e) $\frac{1}{2}x^3$ (f) $\frac{x^6}{3}$ (g) $4x^3 - 2x^2 - 3$ (h) a

2. Integrate with respect to r: (a) $2\pi r$ (b) $4\pi r^2$ (c) 1

3. Find indefinite integrals for: (a) $x(x+2)$ (b) $(2x+1)(x-4)$ (c) $\dfrac{3x^3 - 2x^2 + 3x}{x}$

4. Use your knowledge of negative and fractional indices to find:

(a) $\displaystyle\int \frac{1}{x^2}\,dx$ (b) $\displaystyle\int \frac{1}{4x^3}\,dx$ (c) $\displaystyle\int \sqrt{x}\,dx$

(d) $\displaystyle\int \frac{2}{\sqrt{x}}\,dx$ (e) $\displaystyle\int x\sqrt{x}\,dx$ (f) $\displaystyle\int \sqrt[3]{x}\,dx$

(g) $\displaystyle\int \sqrt[5]{x^3}\,dx$ (h) $\displaystyle\int \frac{2x^3 - 3x^2 - 4}{3x^2}\,dx$ (i) $\displaystyle\int \frac{\sqrt{x} - 1}{2\sqrt{x}}\,dx$

5. Find: (a) $\displaystyle\int dx$ (b) $\displaystyle\int \frac{dx}{\sqrt{x}}$ (c) $\displaystyle\int r\,dr$

6. Find primitives for:

(a) $(3x+2)^3$ (b) $(1-x)^5$ (c) $(x^2+1)^2$ (d) $\dfrac{1}{(x-3)^3}$ (e) $\dfrac{3}{(2x-1)^2}$ (f) $\sqrt{3-2x}$ (g) $\dfrac{2}{3\sqrt{4x+3}}$

7. (a) If $f'(x) = 3x^2$ and $f(1) = -2$, find $f(x)$.

(b) If $\dfrac{dy}{dx} = 4x - 1$ and $y = 7$ when $x = -1$, find y as a function of x.

(c) If $f'(x) = -3$, $f(4) = 7$, find $f(x)$.

(d) If $f'(x) = 12(3x-2)^3$ and the graph of $y = f(x)$ passes through $(1, 2)$, find $f(x)$.

(e) Find the equation of the curve which passes through $(-1, 3)$ and whose gradient function is $\sqrt{2x+11}$.

(f) If $\dfrac{d^2y}{dx^2} = 4x^3 - 1$, and $\dfrac{dy}{dx} = 2$ when $x = 1$, find $\dfrac{dy}{dx}$ as a function of x.

(g) If $\dfrac{d^2y}{dx^2} = 6x + 4$, and $\dfrac{dy}{dx} = 8$ and $y = 2$ when $x = 1$, find y as a function of x.

(h) If $f''(x) = x$, $f'(2) = 1$ and $f(3) = 0$, find $f(x)$.

(i) If $f''(x) = -2x$ and the curve $y = f(x)$ has a stationary point at $(2, 6)$, find the equation of this curve.

Extension

8. A parabola has its vertex at $(1, 2k+1)$ and its second derivative is given by $f''(x) = 2k$.

(a) Find the equation of the parabola.

(b) For what values of k is this parabola negative definite?

9. The velocity v ms^{-1} of a particle moving in a straight line is given by $v = 2t - 1$, where t is the time in seconds.

(a) Find the displacement of the particle as a function of t, if it is initially at $x = -2$.

(b) At what time(s) is the particle at the origin?

(c) When is the particle stationary?

(d) How far does the particle travel in the first 3 seconds?

(e) What is the acceleration of the particle?

10. The acceleration a ms^{-2} of a particle moving along the x axis is given by $\ddot{x} = 1 - t$, where t is the time in seconds. At $t = 1$, the particle is stationary at the origin. Find the displacement of the particle at $t = 3$.

11. A container of water is leaking, so that the rate of change in the volume (V litres) of water is given by $\dfrac{dV}{dt} = t^2 - 36$, where t is the time in seconds after the container is filled.

(a) Find the range of values of t for which $\dfrac{dV}{dt}$ is meaningful, stating how long it takes for the water to stop flowing (i.e. until the container is empty)

(b) Find the amount of water in the container when full.

12. (a) Find the primitive of $x^2 + 2x + 1$.

(b) Without expanding, find the primitive of $(x + 1)^2$, then expand your answer.

(c) Explain why your answers to part (a) and (b) are *not* the same.

13. (a) Differentiate $x(2x + 1)^3$.

(b) Hence find the equation of the curve whose gradient function is $(2x+1)^2(8x+1)$ and passes through the point $(0, 3)$.

14. (a) Differentiate $4x\sqrt{2x + 1}$.

(b) Hence find $\displaystyle\int \frac{3x + 1}{\sqrt{2x + 1}}\,dx$.

Answers to supplementary exercises §1.4.3 on the facing page

1. (a) $\frac{1}{6}x^6$ (b) $\frac{1}{3}x^9$ (c) $\frac{5}{2}x^2$ (d) $7x$ (e) $\frac{1}{8}x^4$ (f) $\frac{1}{21}x^7$ (g) $x^4 - \frac{2}{3}x^3 - 3x$ (h) ax **2.** (a) πr^2 (b) $\frac{4}{3}\pi r^3$ (c) r **3.** (a) $\frac{1}{3}x^3 + x^2$ (b) $\frac{2}{3}x^3 - \frac{7}{2}x^2 - 4x$ (c) $x^3 - x^2 + 3x$ **4.** (a) $-\frac{1}{x} + C$ (b) $-\frac{1}{8x^2} + C$ (c) $\frac{2}{3}\sqrt{x^3} + C$ (d) $4\sqrt{x} + C$ (e) $\frac{2}{5}\sqrt{x^5} + C$ (f) $\frac{3}{4}\sqrt[3]{x^4} + C$ (g) $\frac{5}{8}\sqrt[5]{x^8} + C$ (h) $\frac{1}{3}x^2 - x + \frac{4}{3x} + C$ (i) $\frac{1}{2}x - \sqrt{x} + C$ **5.** (a) x (b) $2\sqrt{x}$ (c) $\frac{1}{2}r^2$ **6.** (a) $\frac{1}{12}(3x + 2)^4 + C$ (b) $-\frac{1}{6}(1 - x)^6 + C$ (c) $\frac{1}{5}x^5 + \frac{2}{3}x^3 + x + C$ (d) $-\frac{1}{2(x-3)^2} + C$ (e) $-\frac{3}{4x-2} + C$ (f) $-\frac{1}{3}\sqrt{(3 - 2x)^3} + C$ (g) $\frac{1}{3}\sqrt{4x + 3} + C$ **7.** (a) $f(x) = x^3 - 3$ (b) $y = 2x^2 - x + 4$ (c) $f(x) = -3x + 19$ (d) $f(x) = (3x-2)^4 + 1$ (e) $y = \frac{1}{3}\sqrt{(2x + 11)^3} - 6$ (f) $\frac{dy}{dx} = x^4 - x + 2$ (g) $y = x^3 + 2x^2 + x - 2$ (h) $f(x) = \frac{1}{6}x^3 - x - \frac{3}{2}$ (i) $y = -\frac{1}{3}x^3 + 4x + \frac{2}{3}$ **8.** (a) $y = kx^2 - 2kx + (3k+1)$ (b) $k < -\frac{1}{2}$ **9.** (a) $x = t^2 - t - 2$ (b) $t = 2$ (c) $t = \frac{1}{2}$ (d) $\frac{13}{2}$ metres (e) 2 ms^{-2} **10.** $x = -\frac{4}{3}$ **11.** (a) $0 \le t \le 6$, 6 seconds (b) 144 L **12.** (a) $\frac{1}{3}x^3 + x^2 + x + C$ (b) $\frac{1}{3}x^3 + x^2 + x + \frac{1}{3} + C$ (c) The constants differ by $\frac{1}{3}$ **13.** (a) $(2x + 1)^2(8x + 1)$ (b) $y = x(2x + 1)^3 + 3$ **14.** (a) $\frac{4(3x+1)}{\sqrt{2x+1}}$ (b) $x\sqrt{2x + 1} + C$

Section 2

Area under a curve & definite integral

Learning Goal(s)

Knowledge
Areas underneath a curve and the x axis

Skills
Finding the areas

Understanding
Why areas between the curve and the x axis can be found via the primitive

By the end of this section am I able to:

21.8 Know that *the area under a curve* refers to the area between a function and the x axis, bounded by two values of the independent variable and interpret the area under a curve in a variety of contexts

21.9 Determine the approximate area under a curve using a variety of shapes including squares, rectangles (inner and outer rectangles), triangles or trapezia

21.12 Using technology or otherwise, investigate the link between the anti-derivative and the area under a curve

21.13 Use the formula $\int_a^b f(x)\,dx = F(b) - F(a)$, where $F(x)$ is the anti-derivative of $f(x)$, to calculate definite integrals

2.1 Approximations of areas beneath a curve by Riemann Sums

Fill in the spaces

- The exact area can be 'sandwiched' (or trapped) between a lower sum and an upper sum.
- Riemann sums
  - Partition the region into shapes (in the Mathematics Advanced course, ... and ....)
  - Taking function values from the .... of a curve and also from the .... of a curve.
  - Sum all of the areas
  - Reduce error by using smaller and smaller shapes

History

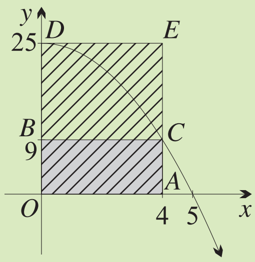
Bernhard Riemann (1826-1866) was a German mathematician who made contributions to analysis, number theory, and differential geometry. In the field of real analysis, he is mostly known for the first rigorous formulation of the integral, the Riemann integral, and his work on Fourier series - crucial for further study in writing functions in terms of periodic/trigonometric functions.

Source: Wikipedia.

Example 24

(Pender et al., 2019b, p.166)

(a) By referring to the diagram, evaluate the bounds on $\int_0^4 \left(25 - x^2\right)\,dx$

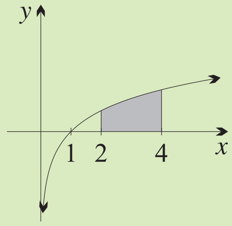


(b) Subdivide the interval $[0, 4]$ as $[0, 2] \cup [2, 4]$ to tighten the bounds.

(c) Subdivide $[0, 4]$ into four subintervals to further tighten the bounds, i.e. to 'sandwich' $\int_0^4 \left(25 - x^2\right)$ between the lower and upper bounds.

Example 25

[Ex 4A Q8/Ex 5A Q10] The area of the region in the diagram is given by $\int_2^4 \ln x \, dx$.



(a) Use 2 lower and 2 upper rectangles to show that (with decimals rounded to 2 places)

$$1.79 < \int_2^4 \ln x \, dx < 2.48$$

(b) Use 4 lower and 4 upper rectangles to show that (with decimals rounded to 2 places)

$$1.98 < \int_2^4 \ln x \, dx < 2.33$$

(c) Use 8 lower and 8 upper rectangles to show that (with decimals rounded to 2 places)

$$2.07 < \int_2^4 \ln x \, dx < 2.24$$

(d) What trend can be identified in the parts above?

GeoGebra

Use this applet to assist: https://www.geogebra.org/m/CfwjsmHx

Further exercises

(A) Ex 4A
- Q6-7, 13

(X1) Ex 5A
- Q8, 9, 11

2.2 Relationship between area beneath curve & the primitive

Steps

1. $A(x)$: area between the curve f and the x axis from a to x.

2. Lower/upper sum.

- Under/over estimation.
- Take more rectangles.

3. $\lim_{\Delta x \to 0} \left(\sum_{x=a}^{b} f(x) \Delta x \right)$.

4. Sandwich theorem.

5. Evaluate $F(x)$ at $x = a$.

Theorem 6

(First) Fundamental Theorem of Calculus If $f(x)$ is continuous over the closed interval $[a, b]$ and F is the primitive of $f(x)$, then

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

Differential form:

$$\frac{d}{dx}\int_a^x f(t)\,dt = \ldots\ldots\ldots\ldots$$

Example 26

[Ex 5D] Use the differential form of the fundamental theorem to simplify these expressions. Then confirm the consistency of the discussion in this section by performing the integration and then differentiating.

(a) $\dfrac{d}{dx}\displaystyle\int_1^x 6t^2\,dt$ (b) $\dfrac{d}{dx}\displaystyle\int_{-2}^x \left(t^3 - 9t^2 + 5\right)dt$ (c) $\dfrac{d}{dx}\displaystyle\int_4^x \frac{1}{t^2}\,dt$

Further exercises

Ⓐ **Ex 4D**

- All questions

ⓧ1 **Ex 5D**

- All questions

2.3 Evaluating definite integrals

Definition 2

- $\int_a^b f(x)\,dx$ is a .... integral.
- $\int f(x)\,dx$ is an .... integral.

Important note

- .... integrals are *evaluated*. **Keyword:** ....
- .... integrals are *found*. **Keyword:** ....

Example 27

Find the area beneath the curve $y = x^2$ from $x = 0$ to $x = 3$. **Answer:** 9

Example 28

Evaluate $\displaystyle\int_2^3 \left(x^3 - x^2 + 1\right)\,dx$. **Answer:** $\frac{131}{12}$

2.3.1 Supplementary exercises

1. Evaluate the following definite integrals:

(a) $\displaystyle\int_0^1 x^3\,dx$

(b) $\displaystyle\int_0^2 \left(3x^2 - x + 1\right)\,dx$

(c) $\displaystyle\int_1^2 \left(x^3 + 1\right)\,dx$

(d) $\displaystyle\int_1^4 \sqrt{x}\,dx$

(e) $\displaystyle\int_2^3 \frac{dx}{x^2}$

(f) $\displaystyle\int_{-3}^3 \left(1 - x^2\right)\,dx$

(g) $\displaystyle\int_0^1 (2x+1)^3\,dx$

(h) $\displaystyle\int_{-4}^0 \sqrt{1-2x}\,dx$

(i) $\displaystyle\int_1^2 \frac{3x^3 - 2x^2 + 1}{3x^2}\,dx$

(j) $\displaystyle\int_1^2 \left(x + \frac{1}{x}\right)^2\,dx$

(k) $\displaystyle\int_{-2}^3 dx$

(l) $\displaystyle\int_1^4 x\sqrt{x}\,dx$

(m) $\displaystyle\int_4^9 \frac{1}{2\sqrt{x}}\,dx$

(n) $\displaystyle\int_1^{14} \frac{1}{\sqrt[3]{2x-1}}\,dx$

(o) $\displaystyle\int_0^1 \frac{x^2 + 2x + 2}{(x+1)^2}\,dx$

2. (a) Solve for k: $\displaystyle\int_1^3 kx^2\,dx = \frac{65}{3}$.

(b) Solve for x: $\displaystyle\int_1^x \frac{dt}{\sqrt{t}} = 1$.

Answers to supplementary exercises §2.3.1

1. (a) $\frac{1}{4}$ (b) 8 (c) $\frac{19}{4}$ (d) $\frac{14}{3}$ (e) $\frac{1}{6}$ (f) -12 (g) 10 (h) $\frac{26}{3}$ (i) 1 (j) $\frac{29}{6}$ (k) 5 (l) $\frac{62}{5}$ (m) 1 (n) 6 (o) $\frac{3}{2}$ **2.** (a) $k = \frac{5}{2}$ (b) $x = \frac{9}{4}$

Further exercises

Ⓐ **Ex 4B**
- All questions

Ex 4E
- Q10, 15, 16, 19

Ex 4I
- Q15

ⓍⅠ (X1) **Ex 5B**
- Q1-16 last 2 columns

Ex 5E
- Q10, 15, 16, 19
- (X2) Q20

Ex 5I
- Q15
- (X2) Q16

Further exercises (Legacy Textbooks)

Ex 11B Q1-6

Ex 11D Q7

Section 3

Properties of definite integrals

Learning Goal(s)

Knowledge
Difference between definite and indefinite integral, and signed areas

Skills
Finding definite integrals

Understanding
When to apply the 'straight integral' versus finding the area enclosed

By the end of this section am I able to:

21.11 Understand the relationship of position to signed areas, namely that the signed area above the horizontal axis is positive and the signed area below the horizontal axis is negative

21.14 Use geometric ideas to find the definite integral $\int_a^b f(x)\,dx$ where $f(x)$ is positive throughout an interval $a \leq x \leq b$ and the shape of $f(x)$ allows such calculations, for example when $f(x)$ is a straight line in the interval or $f(x)$ is a semicircle in the interval.

21.15 Calculate the area under a curve

21.16 Calculate areas between curves determined by any functions within the scope of this syllabus

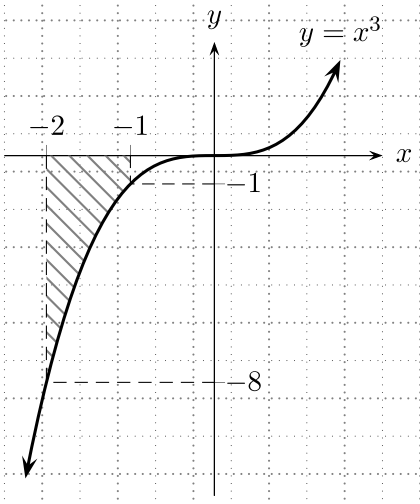
3.1 Area above/below x axis

Definition 3

- Evaluation of definite integrals result in .... areas.
- Area above x axis is .... ; area beneath is ....

Example 29

Evaluate $\int_{-2}^{-1} x^3\,dx$. **Answer:** $-\frac{15}{4}$



y
y = x^3
−2
−1
x
−1
−8

3.2 Simple geometry

- Some integrals may be evaluated by sketching a diagram; which would otherwise be very difficult to evaluate.
- Some other questions only provide a diagram and the integral is to be evaluated graphically.

Example 30

Evaluate $\displaystyle\int_0^3 \sqrt{9-x^2}\,dx$. **Answer:** $\frac{9\pi}{4}$

Example 31

[2016 2U Q9] What is the value of $\displaystyle\int_{-3}^{2} |x+1|\,dx$?

(A) $\dfrac{5}{2}$ (B) $\dfrac{11}{2}$ (C) $\dfrac{13}{2}$ (D) $\dfrac{17}{2}$

Important note

Draw picture!

Further exercises

(A) Ex 4A
- Q4-5, 9-10

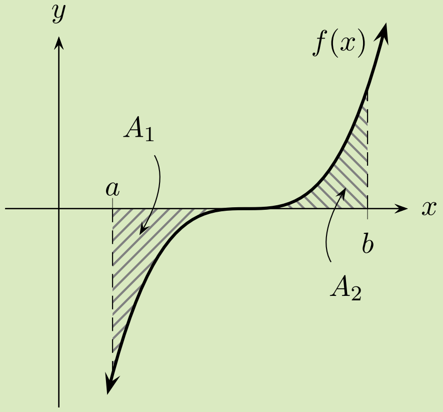
(x1) Ex 5A
- Q4-7

3.3 By diagrams

- These integrals can only be evaluated graphically.
- Be aware of areas above or below the x axis, as well as any signed 'areas'.

Example 32

The areas A_1 and A_2 are bounded by the curve $f(x)$, the x axis, the line $x = a$ and $x = b$. The areas are 25 and 15 sq. units respectively.

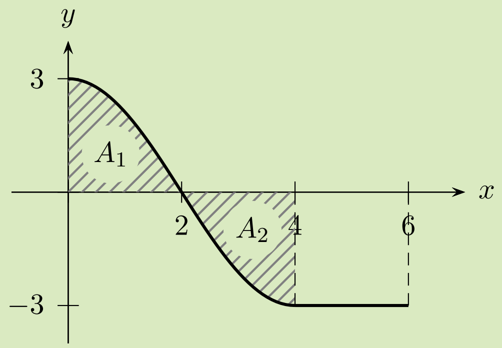


Evaluate the integral $\displaystyle\int_a^b f(x)\,dx$, justifying your answer. **Answer:** -10

Example 33

[2010 2U HSC Q9(b)] Let $y = f(x)$ be a function defined for $0 \le x \le 6$, with $f(0) = 0$.

The diagram shows the graph of the derivative of f, $y = f'(x)$.



The shaded region A_1 has area 4 square units. The shaded region A_2 has the area 4 square units.

(i) For which values of x is $f(x)$ increasing? **1**

(ii) What is the maximum value of $f(x)$? **1**

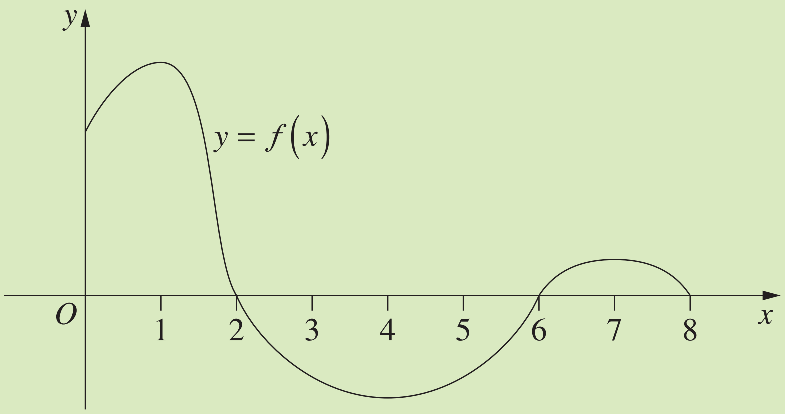
(iii) Find the value of $f(6)$. **1**

(iv) Draw a graph of $y = f(x)$, for $0 \le x \le 6$. **2**

Answer: i. $0 \le x < 2$. ii. 4. iii. -6. iv. $f(4) = 0$, $f(6) = -6$. Local max at $x = 2$.

Example 34

[2012 2U HSC Q10] The graph $y = f(x)$ has been drawn to scale for $0 \le x \le 8$.

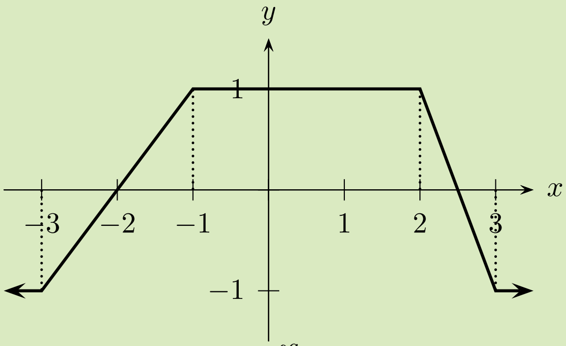


Which of the following integrals has the greatest value?

(A) $\displaystyle\int_0^1 f(x)\,dx$ (B) $\displaystyle\int_0^2 f(x)\,dx$ (C) $\displaystyle\int_0^7 f(x)\,dx$ (D) $\displaystyle\int_0^8 f(x)\,dx$

Example 35

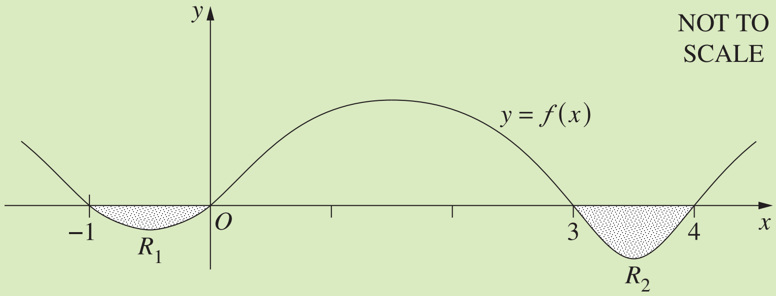
[2013 2U HSC Q14] The diagram shows $y = f(x)$.



What is the value of a, where $a > 0$ so that $\displaystyle\int_{-a}^{a} f(x)\,dx = 0$?

Example 36

[2018 2U HSC Q7] The diagram shows the graph of $y = f(x)$ with intercepts at $x = -1, 0, 3$ and 4.



The area of shaded region R_1 is 2. The area of shaded region R_2 is 3.

It is given that $\int_0^4 f(x)\,dx = 10$.

What is the value of $\int_{-1}^{3} f(x)\,dx$?

(A) 5 (B) 9 (C) 11 (D) 15

3.4 Symmetry

3.4.1 Odd functions

Definition 4

A .... is an interval with 0 as the centre, and spread out equally on either side of zero.

Theorem 7

The integral of an *odd function* over a .... is zero, i.e.

$$\int_{-a}^{a} f(x)\,dx = 0$$

if $f(x)$ is odd.

Example 37

Evaluate $\displaystyle\int_{-1}^{1} \left(5x^5 + x^3 - x\right)\,dx$.

3.4.2 Even functions

Theorem 8

The integral of an *even function* over a .... is double the integral over half the interval, i.e.

$$\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx$$

if $f(x)$ is even.

Example 38

Evaluate $\displaystyle\int_{-2}^{2} \left(x^2 - 4\right)\,dx$.

3.5 Reversal of limits

Theorem 9

An integral with the limits reversed, is the negative of the original integral, i.e.

$$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$$

Proof

Steps

1. $\displaystyle\int_a^b f(x)\,dx =$ ....

2. $\displaystyle\int_b^a f(x)\,dx =$ .... $=$ ....

3.6 Greater/lesser y values

Theorem 10

If $x \in [a, b]$, $f(x) \le g(x)$, then

$$\int_a^b f(x)\, dx \le \int_a^b g(x)\, dx$$

Example 39

(a) Sketch the graph of $f(x) = 4 - x^2$ for $x \in [-2, 2]$.

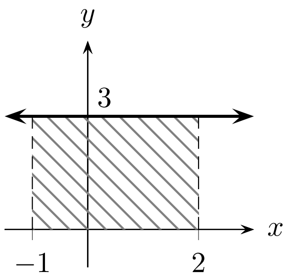
(b) Explain why $0 \le \displaystyle\int_{-2}^{2} \left(4 - x^2\right)\, dx \le 16$.

Supplementary exercises

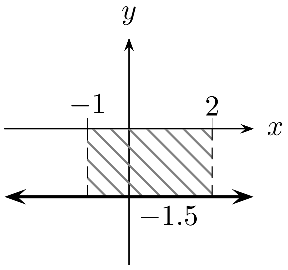
1. Find in each case, the

i. shaded area ii. value of $\int_a^b f(x)\,dx$ where a and b have values shown

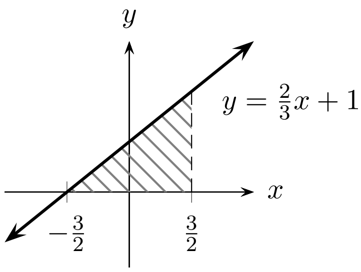
(a)



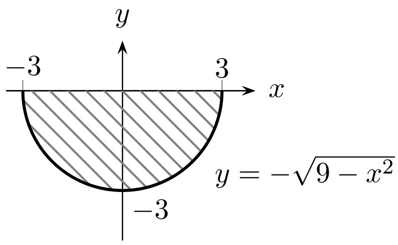
(b)



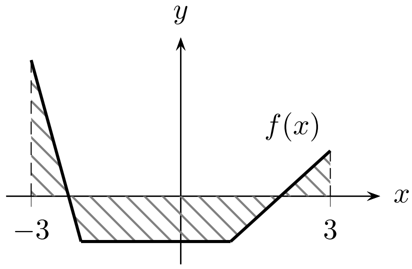
(c)



(d)



(e)



$$f(x) = \begin{cases} -4x - 9 & x < -2 \\ -1 & -2 \le x < 1 \\ x - 2 & x \ge 1 \end{cases}$$

2. By drawing appropriate diagrams, evaluate:

(a) $\int_{-2}^{4} (x+2)\,dx$

(b) $\int_{2}^{5} (4-2x)\,dx$

(c) $\int_{-1}^{3} (-x)\,dx$

(d) $\int_{-3}^{5} dx$

(e) $\int_{0}^{6} (4+3x)\,dx$

(f) $\int_{-5}^{5} \sqrt{25-x^2}\,dx$

(g) $\int_{0}^{1} \sqrt{4-y^2}\,dy$

(h) $\int_{5}^{1} \frac{x}{2}\,dx$

(i) $\int_{0}^{1} (1-2y)\,dy$

3. If $f(x) = \begin{cases} -1 & x < -2 \\ x+1 & -2 \le x < 0 \\ \sqrt{1-x^2} & 0 \le x \\ 0 & x \ge 1 \end{cases}$, evaluate $\int_{-4}^{4} f(x)\,dx$.

4. (a) On a number plane, draw the line segment $2x + 3y - 18 = 0$, $3 < x < 6$.

(b) *Without* calculating, write down in integral form

i. the area of the region bounded by the given line segment, the x axis, and the vertical lines joining the end of this line segment to the x axis.

ii. the area of the region bounded by the given line segment, the y axis, and the horizontal lines joining the end of this line segment to the y axis.

5. (a) On a number plane, draw the lines $y = 2x$ and $x + 2y - 10 = 0$, showing their intercepts with the axes and the point of intersection. Shade in the region bounded by the two lines and the x axis.

(b) *Without* calculating the area, write down, in integral form, the area of the shaded region.

6. (a) On a number plane, draw the parabola $y = 3x - x^2$ and the line $y = x$, showing the intercepts with the axes, and the points of intersection. Shade in the region bounded by the parabola and the line.

(b) Write down, in integral form, the area of the shaded region.

7. (a) i. Draw a sketch of $y = x^3$ ii. Hence evaluate $\displaystyle\int_{-2}^{2} x^3\,dx$.

(b) i. Show that $f(x) = \dfrac{2x}{1+x^2}$ is an odd function. ii. Hence evaluate $\displaystyle\int_{-3}^{3} \frac{2x}{1+x^2}\,dx$.

(c) i. Show that $f(x) = x^4 - x^2 + 1$ is an even function.

ii. Hence evaluate $\displaystyle\int_{-5}^{5} \left(x^4 - x^2 + 1\right)\,dx$.

Extension

8. If $f(x)$ is continuous $\forall x$, $f'(x) = 0\ \forall x$ and $f(3) = 4$, find $\displaystyle\int_{-1}^{3} f(x)\,dx$.

9. (a) By drawing the graph of $y = x^2$, state why $\displaystyle\int_{0}^{2} x^2\,dx + \int_{0}^{4} \sqrt{y}\,dy = 8$.

(b) i. Sketch a graph of $y = \log_2 x$.

ii. It is given that $\displaystyle\int_{0}^{2} 2^y\,dy = k$, where k is a constant. Use your graph and technique in part (a) to find an expression for $\displaystyle\int_{1}^{4} \log_2 x\,dx$ in terms of k.

10. (a) By drawing the graph of $y = 2t$, find as functions of x:

i. $\displaystyle\int_{0}^{x} 2t\,dt$. ii. $\displaystyle\int_{2}^{x} 2t\,dt$

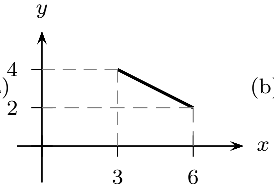
(b) i. Use part (a) (ii) to solve the equation $\displaystyle\int_{2}^{x} 2t\,dt = 32$.

ii. Explain in terms of areas how there can be two answers to this equation.

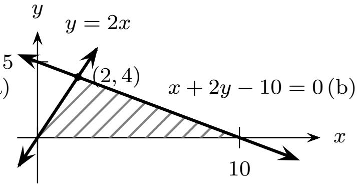
Answers to supplementary exercises §3.6 on page 39

1. (a) i. 9 units^2 ii. 9 (b) i. 4.5 units^2 ii. -4.5 (c) i. 3 units^2 ii. 3 (d) i. $\frac{9\pi}{2}$ units^2 ii. $-\frac{9\pi}{2}$ (e) i. $\frac{21}{4}$ units^2 ii. -2 **2.** (a) 18 (b) -9 (c) -4 (d) 8 (e) 78 (f) $\frac{25\pi}{2}$ (g) $\frac{3\sqrt{3}+2\pi}{6}$ (h) -6 (i) 0 **3.** $\frac{\pi}{4} - 2$

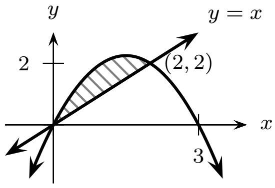
4. (a)



(b) i. $\int_3^6 \left(6 - \frac{2}{3}x\right)\,dx$ ii. $\int_2^4 \left(9 - \frac{3}{2}y\right)\,dy$ **5.** (a)

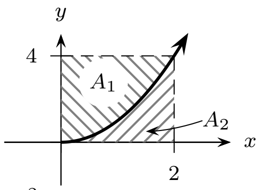


(b) $\int_0^2 2x\,dx + \int_2^{10} \left(5 - \frac{x}{2}\right)\,dx$ **6.** (a)



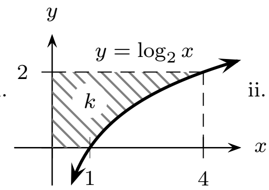
(b) $\int_0^2 \left(3x - x^2\right)\,dx - \int_0^2 x\,dx$ **7.** (a) 0 (b) 0 (c) $2\int_0^5 \left(x^4 - x^2 + 1\right)\,dx$

8. 16 **9.** (a)



$$A_1 = \int_0^4 \sqrt{y}\,dx$$
$$A_2 = \int_0^2 x^2\,dx$$
$$A_1 + A_2 = 4 \times 2 = 8$$

(b) i.



ii. $\int_1^4 \log_2 x\,dx = 8 - k$ **10.** (a) i. x^2 ii. $x^2 - 4$ iii. $x^2 - 1$ (b) $x = \pm 6$

Further exercises

(A) Ex 4C
- All questions

(X1) Ex 5C
- Q9-18

Further exercises (Legacy Textbooks)

Ex 11C all questions

Ex 11H Q8-10

Section 4

Finding areas via integration

4.1 Area via diagram

Fill in the spaces

- The integral $\int_a^b f(x)\,dx$ may be less than the area between the curve $f(x)$ and the x axis if $f(x)$ has .... between $a \leq x \leq b$.
- Draw picture!
- Use .... on integrals as appropriate.

Important note

Key phrase: area .... /area .... by. Beware whenever the diagram shows the curve crossing the x axis, or where the x intercepts need to be found.

Example 40

Find the area enclosed by the curve $y = x^3$, the x axis and the lines $x = -1$ and $x = 2$.

Answer: $\frac{17}{4}$

Example 41

Find the area enclosed by the curve $y = x(x+1)(x-2)$ and the x axis. **Answer:** $\frac{37}{12}$

Example 42

Find the area bounded by the curve $y = 9x - x^3$ and the x axis. **Answer:** $\frac{81}{2}$

Further exercises

Ⓐ **Ex 4F**

- Q1, 6-9

(x1) **Ex 5F**

- Q1, 3, 5-8

4.2 (X1) Area between curve and y axis

Important note

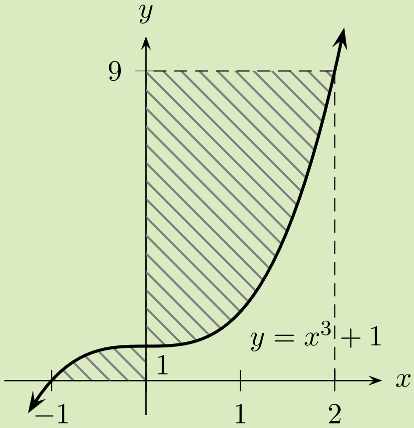
⚠ This section may or may not be in the Mathematics Advanced syllabus but is implicitly required for future Mathematics Extension 1 work. At the time of printing, no confirmation has been received from NESA whether it's present or not.

Laws/Results

- The area to the right of the y axis is .... ; area to the left is ....
- Change the .... from x to y where necessary.

Example 43

Find the area of the region enclosed, given the curve is $y = x^3 + 1$. **Answer:** $\frac{51}{4}$

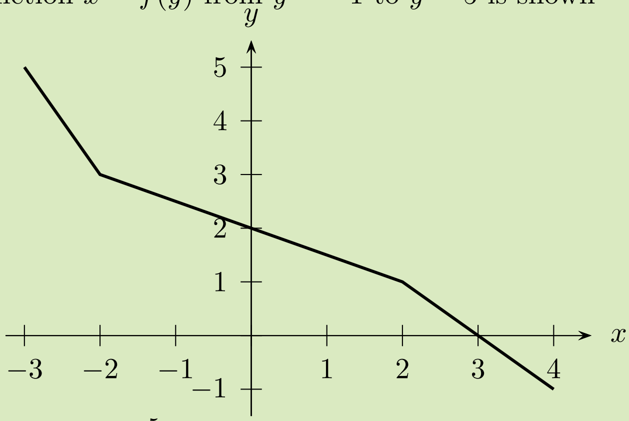


Example 44

Find the area bounded by the coordinate axes and the curve $y^2 = 16(2-x)$. **Answer:** $\frac{32}{3}\sqrt{2}$

Example 45

The graph of the function $x = f(y)$ from $y = -1$ to $y = 5$ is shown



Use this diagram to evaluate $\displaystyle\int_{-1}^{5} f(y)\,dy$.

Further exercises

(A) **Ex 4F**
- Q2, 3, 5

(X1) **Ex 5F**
- Q2, 4

4.2.1 Supplementary exercises

1. Sketch the region bounded by the given curve and the given lines. Find the area of this region.

(a) $y = x^2 + 3$, the x axis, and the lines $x = 1$ and $x = 3$.

(b) The part of $y = -x(x-1)(x-2)$ below the x axis, and the x axis.

(c) $x = y^3$, the y axis and the line $y = 2$.

(d) $y = -x^2 + 5x - 4$ and the x axis.

(e) $x = y^2 - y$ and the y axis.

(f) $y = \dfrac{1}{x^2}$, the x axis and the lines $x = 1$ and $x = 4$.

(g) $y = (3x-2)^3$, the x axis and the y axis.

(h) $x = \sqrt{1-y}$ and the coordinate axes.

(i) $x = -\sqrt{36 - y^2}$ and the y axis.

2. Shade the region bounded by the given curve and the given lines. By first changing the subject of the equation, find the shaded area.

(a) $y = \sqrt{2x}$, the y axis and the line $y = 4$.

(b) $y = x^2 + 1$, and the line $y = 5$.

Extension

3. (a) Differentiate $\left(x^3+1\right)^5$. (b) Hence evaluate $\displaystyle\int_0^1 x^2\left(x^3+1\right)^4\,dx$.

4. (a) Differentiate $\sqrt{1+2x^2}$.

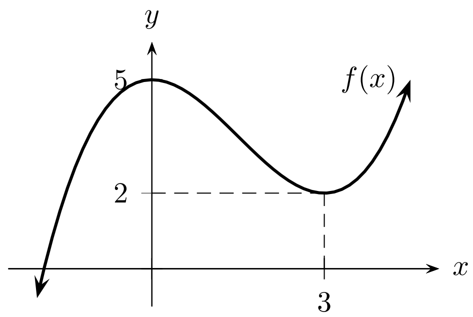
(b) Hence find the area bounded by the curve $x = \dfrac{3y}{\sqrt{1+2y^2}}$, the y axis & the lines $y = 2$ and $y = 12$.

5. The graph of $y = f(x)$ has a local maximum at $(0, 5)$ and a local minimum at $(3, 2)$.

(a) Copy the graph of $y = f(x)$, and on another set of axes directly below it, sketch the graph of $y = f'(x)$.

(b) On your diagram, shade the region bounded by $y = f'(x)$ and the x axis.

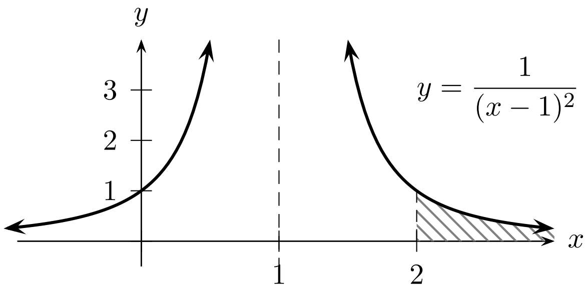
(c) Find the area of the shaded region.



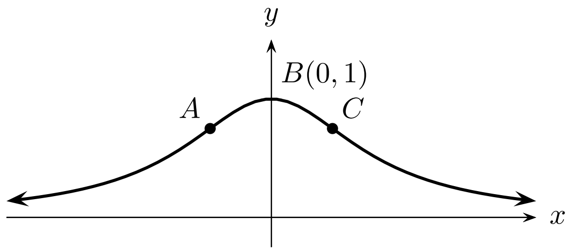
6. (a) i. Find $\displaystyle\int_1^x \frac{dt}{t^3}$.

ii. Hence find $\displaystyle\lim_{x\to\infty}\int_1^x \frac{dt}{t^3}$.

(b) The graph shows the region 'bounded' by the curve $y = \dfrac{1}{(x-1)^2}$ and the line $x = 2$. Find the area of this region.



7. The diagram shows the graph of $y = f(x)$. $B(0,1)$ is a local maximum. A and C are points of inflexion. The x axis is an asymptote of the curve.



(a) At which point on the curve does the tangent have its maximum gradient?

(b) Copy the diagram. On a separate set of axes directly below your sketch, draw a sketch of $y = f'(x)$, the gradient function. Line up the features corresponding to points A, B and C.

(c) Calculate the total area bounded by the curve $y = f'(x)$ and the x axis.

8. The velocity, v ms^{-1}, of a particle t seconds after starting to move is given by $\dot{x} = t(3-t)$. Sketch the graph of the velocity function. Find:

(a) the distance travelled by the particle in the first three seconds.

(b) the distance travelled in the next three seconds.

(c) the total distance travelled by the particle in the first six seconds.

(d) the displacement of the particle from its initial position after six seconds.

9. (a) Sketch the curve $y = x^4$ and shade the region bounded by this curve, the x axis, and the line $x = 2$.

(b) Find the area of the shaded region.

(c) Without further integration, find the area bounded by this curve, the line $y = 16$, and the y axis, in the first quadrant.

10. (a) On a number plane, sketch the curve $y = 4x^2 + 4x + 2$, showing the coordinates of the vertex. Shade the region bounded by this curve, the y axis, and the line $y = 10$, in the first quadrant.

(b) By writing the above equation as $4x^2 + 4x + (2 - y) = 0$, and using the quadratic formula, make x the subject of this formula.

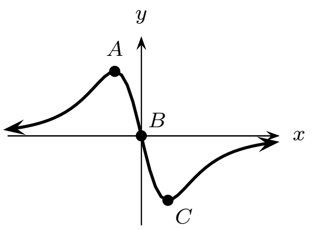
(c) Hence, find the area of the shaded region.

(d) How could this area be calculated without changing the subject of the equation? Perform this calculation.

Answers to supplementary exercises §4.2.1 on page 47

1. (a) $\frac{44}{3}$ units^2 (b) $\frac{1}{4}$ units^2 (c) 4 units^2 (d) $\frac{9}{2}$ units^2 (e) $\frac{1}{6}$ units^2 (f) $\frac{3}{4}$ units^2 (g) $\frac{4}{3}$ units^2 (h) $\frac{2}{3}$ units^2 (i) 18π units^2 **2.** (a) $\frac{32}{3}$ units^2 (b) $\frac{32}{3}$ units^2 **3.** (a) $15x^2\left(x^3+1\right)^4$ (b) $\frac{31}{15}$ **4.** (a) $\frac{2x}{\sqrt{1+2x^2}}$ (b) 21 units^2 **5.** 3 units^2 **6.** (a) i. $\frac{1}{2}\left(1 - \frac{1}{x^2}\right)$ ii. $\frac{1}{2}$ (b) 1 units^2

7. (a) A (b)



(c) 2 units^2 **8.** (a) $\frac{9}{2}$ m (b) $\frac{45}{2}$ m (c) 27 m (d) -18 m **9.** (a) Not sketched here. (b) $\frac{32}{5}$ units^2 (c) $\frac{128}{5}$ units^2 **10.** (a) parabola, concave up, $V\left(-\frac{1}{2}, 1\right)$, y intercept $= 2$ (b) $x = \frac{-1 \pm \sqrt{y-1}}{2}$ (c) $\frac{14}{3}$ units^2

Further exercises (Legacy Textbooks)

Ex 11E Q1-16

Section 5

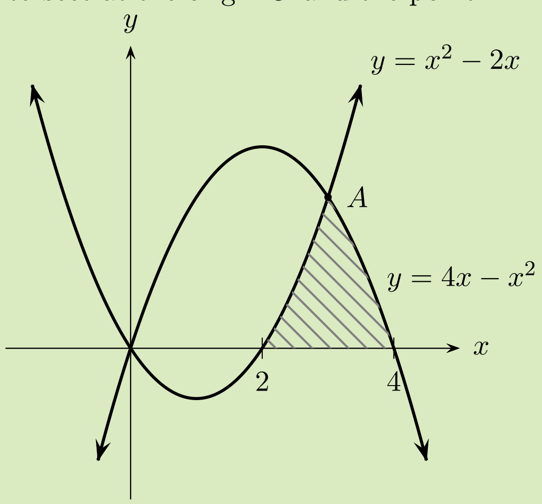
Further areas

5.1 Compound areas by addition

- Some areas to be found are composed of the area between the one curve and the x axis, plus another curve and the x axis.

Example 46

[**2016 Independent Q13**] The diagram below shows the parabolas $y = 4x - x^2$ and $y = x^2 - 2x$. The graphs intersect at the origin O and the point A. **Answer:** 3



(i) Find the x coordinate of the point A. **1**

(ii) Find the area of the shaded region bounded by the two parabolas and the x axis. **3**

5.2 Compound areas by subtraction

Laws/Results

If $f(x) \leq g(x)$ over the interval $a \leq x \leq b$, then the area between the curves $f(x)$ and $g(x)$ is given by

$$A = \underbrace{\ldots\ldots\ldots\ldots\ldots\ldots\ldots}_{\text{top area}} - \underbrace{\ldots\ldots\ldots\ldots\ldots\ldots\ldots}_{\text{bottom area}}$$

$$= \int_a^b \left(g(x) - f(x)\right)\, dx$$

Fill in the spaces

- Find .... between $f(x)$ and $g(x)$.
- If a sketch of the graph(s) is not possible, apply the .... :

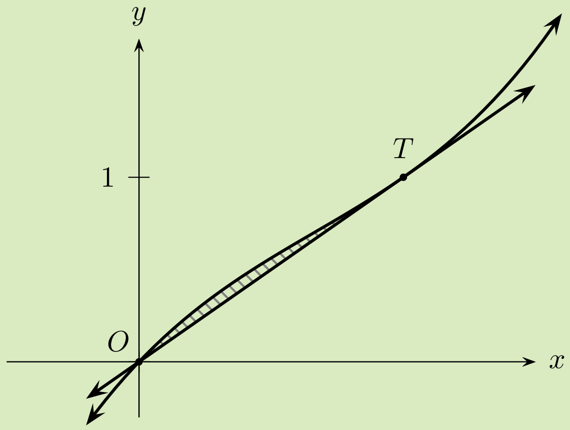
$$A = \left| \int_a^b g(x) - f(x)\, dx \right|$$

Example 47

Find the area bounded by the curves $f(x) = 2x - x^2$ and $g(x) = 2 - x$. **Answer:** $\frac{1}{6}$

Example 48

[2013 2U HSC Q13] The diagram shows the graphs of the functions $f(x) = 4x^3 - 4x^2 + 3x$ and $g(x) = 2x$. The graphs meet at O and T. **Answer:** i. $x = 0.5$ ii. $\frac{1}{48}$



(i) Find the x coordinate of T. **1**

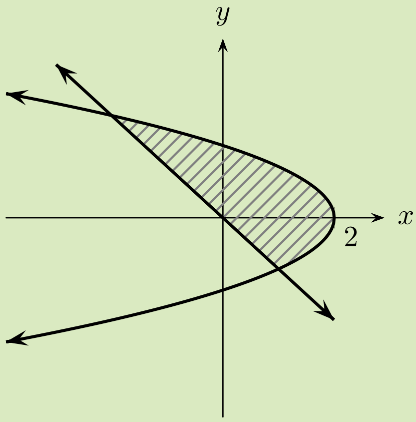
(ii) Find the area of the shaded region between the graphs of the functions $f(x)$ and $g(x)$. **3**

Example 49

Find the area bounded by the curves $y = \sqrt{x}$ and $y = x^2$. **Answer:** $\frac{1}{3}$

Example 50

(X1) Find the region bounded by $y^2 = -(x-2)$ and $y = -x$. **Answer:** $\frac{9}{2}$



Example 51

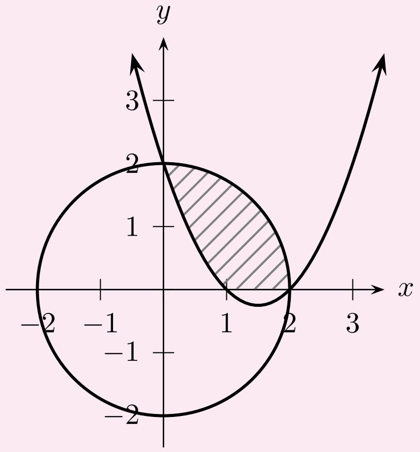
Find the area bounded by $f(x) = x^3 - 2x^2 - x + 2$ and $g(x) = x^2 - 1$. **Answer:** 8

Important note

Draw this diagram carefully!

Timed exam practice 1 (Allow approximately 5 minutes)

[**2005 2U HSC Q8**] (3 marks) The shaded region in the diagram is bounded by the circle of radius 2 centred at the origin, the parabola $y = x^2 - 3x + 2$, and the x axis.



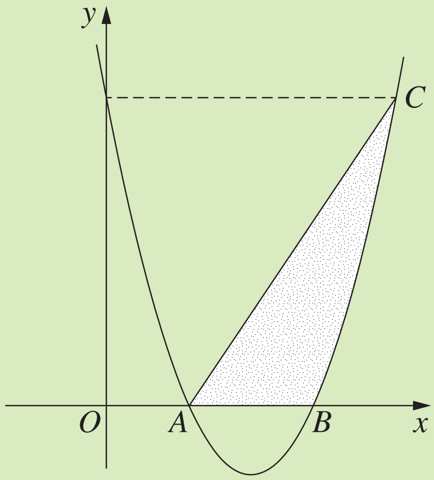
By considering the difference of two areas, find the area of the shaded region.

Marking criteria

- ✓ [1] Finds the area of the quadrant of the circle or writes down a correct integral expression for the area or equivalent merit
- ✓ [2] Finds a strategy involving two or more areas and correctly calculates one of these areas or equivalent merit
- ✓ [3] Correct solution of $\pi - \frac{5}{6}$

Example 52

[2015 2U HSC Q15] The diagram shows the curve with equation $y = x^2 - 7x + 10$. The curve intersects the x-axis at points A and B. The point C on the curve has the same y-coordinate as the y-intercept of the curve.



(i) Find the coordinate of points A and B. **1**

(ii) Write down the coordinates of C. **1**

(iii) Evaluate $\int_0^2 \left(x^2 - 7x + 10\right)\, dx$. **1**

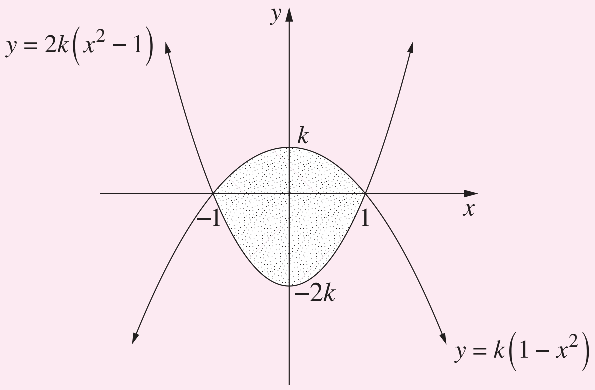
(iv) Hence or otherwise, find the area of the shaded region. **2**

Answer: $\frac{49}{3}$

Timed exam practice 2 (Allow approximately 5 minutes)

[2017 2U HSC Q14] (3 marks) The shaded region shown is enclosed by two parabolas, each with x intercepts at $x = -1$ and $x = 1$.

The parabolas have equations $y = 2k\left(x^2 - 1\right)$ and $y = k\left(1 - x^2\right)$, where $k > 0$.



Given that the area of the shaded region is 8, find the value of k.

Marking criteria

- ✓ [1] Attempts to use integration to find the area of the shaded region, or equivalent merit
- ✓ [2] Obtains an expression for the area involving k with integration complete, or equivalent merit
- ✓ [3] Provides correct solution

Further exercises

(A) Ex 4G

- All questions

(x1) Ex 5G

- All questions

5.2.1 Supplementary exercises

1. Find the area of the region bounded by the given curve and the given lines:

(a) $y = 9 - x^2$, the x axis, and the lines $x = 0$ and $x = 5$.

(b) $y = x^3 - x$ and the x axis.

(c) $x = y^2 - 3y$, the y axis, and the lines $y = -2$ and $y = 2$.

(d) $y = \left|x^2 - 3x + 2\right|$ and the x axis, between $x = 0$ and $x = 3$.

2. Sketch the region bounded by the following pair of curves. Find the area of this region.

(a) $y = x^2$ and $y = 2x$

(b) $y = x^4$ and $y = x^2$

(c) $y = 2x - x^2$ and $y = 2 - x$

(d) $y = 4 - x^2$ and $y = x^2$

(e) $f(x) = 3x^2$ and $g(x) = 4x - x^2$

(f) $y = \sqrt{x}$, $y = 6 - x$ and the x axis

(g) $x^2 = 4y$ and $y^2 = 4x$

(h) $x = 4 - y$ and $x = 5y - y^2 - 4$

(i) $y = x^3 - 2x^2 - x + 2$ and $y = x^2 - 1$

(j) $y = x^2(2 - x)$ and $y = x(2 - x)^2$

3. Find the area of the region bounded by

(a) $y = 9 - x^2$, $y = 1 - x^2$ and the x axis.

(b) $y = x^2$, $y = (x - 4)^2$ and the x axis.

(c) $y = -x^2 + 4x - 3$, the y axis and the lines $x = 4$ and $y = 4$.

4. Find the area of the region $\left\{y \leq 4 - x^2\right\} \cap \left\{y = x^2 - 4\right\}$.

Extension

5. (a) Prove that the line $y = x + 2$ is a tangent to the parabola $y = x^2 - 5x + 11$.

(b) Let Q be the point where the line $y = x + 2$ touches the parabola $y = x^2 - 5x + 11$. Find the area of the region enclosed between the parabola and the normal to the parabola at Q.

6. Sketch the region in the Cartesian plane bounded by the parabola $y = x^2 - 4x + 3$, its tangent at $(3, 0)$ and its axis of symmetry. Find the area of this region.

Answers to supplementary exercises §5.2.1

1. (a) $\frac{98}{3}$ units^2 (b) $\frac{1}{2}$ units^2 (c) 12 units^2 (d) $\frac{11}{6}$ units^2 **2.** (a) $\frac{4}{3}$ units^2 (b) $\frac{4}{15}$ units^2 (c) $\frac{1}{6}$ units^2 (d) $\frac{16\sqrt{2}}{3}$ units^2 (e) $\frac{2}{3}$ units^2 (f) $\frac{22}{3}$ units^2 (g) $\frac{16}{3}$ units^2 (h) $\frac{4}{3}$ units^2 (i) 8 units^2 (j) 1 unit^2 **3.** (a) $\frac{104}{3}$ units^2 (b) $\frac{16}{3}$ units^2 (c) $\frac{52}{3}$ units^2 **4.** $\frac{64}{3}$ units^2 **5.** $\frac{4}{3}$ units^2 **6.** $\frac{1}{3}$ units^2

Further exercises (Legacy Textbooks)

Ex 11F Q1-15

Section 6

Approximating the definite integral

Learning Goal(s)

Knowledge
Iterative nature of applying the trapezoidal rule

Skills
Applying the trapezoidal rule

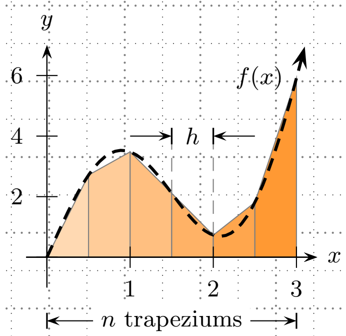
Understanding
When to use the trapezoidal rule

By the end of this section am I able to:

21.17 Use the Trapezoidal rule to estimate areas under curves

6.1 Trapezoidal rule

- Trapeziums are a better approximation to the exact area bounded by a curve, the x-axis within the interval $x \in [a, b]$.
- Divide the curve into n trapeziums,



$$A = \frac{1}{2}h(a+b)$$

- 1st trapezium

$$A_1 = \frac{1}{2}h\left(f(0) + f\left(\frac{1}{2}\right)\right)$$

- 2nd trapezium

$$A_2 = \frac{1}{2}h\left(f\left(\frac{1}{2}\right) + f(1)\right)$$

- 3rd trapezium

$$A_3 = \frac{1}{2}h\left(f(1) + f\left(\frac{3}{2}\right)\right)$$

- Generalising,

Laws/Results

Trapezoidal rule:

$$A \approx \frac{h}{2}\left(y_1 + 2\left(\sum y_{\text{middle}}\right) + y_\ell\right)$$

where h is the distance between two function values, y_ℓ being the last function value.

Example 53

Find the approximate area under the curve $f(x) = \frac{1}{x}$ using the Trapezoidal Rule with 3 function values from $x = 1$ to $x = 2$. **Answer:** 0.708

Steps

1. Draw table of values:

x	1	$\frac{3}{2}$	2
y			

2. Apply Trapezoidal Rule:

Example 54

Use the Trapezoidal Rule with five function values to approximate

$$\int_1^3 \log_{10} x \, dx$$

Express your answer correct to 4 decimal places. **Answer:** 0.5568

Example 55

[1996 2U HSC Q4] Let $f(x) = \sqrt{25 - x^2}$.

i. Supply the missing values in this table: **3**

x	0	1	2	3	4	5
$f(x)$	5.000		4.583			0.000

ii. Use these six values of the function and the trapezoidal rule to find the approximate value of **3**

$$\int_0^5 \sqrt{25 - x^2}\, dx$$

iii. Draw the graph of $x^2 + y^2 = 25$ and shade the region whose area is represented by the integral **1**

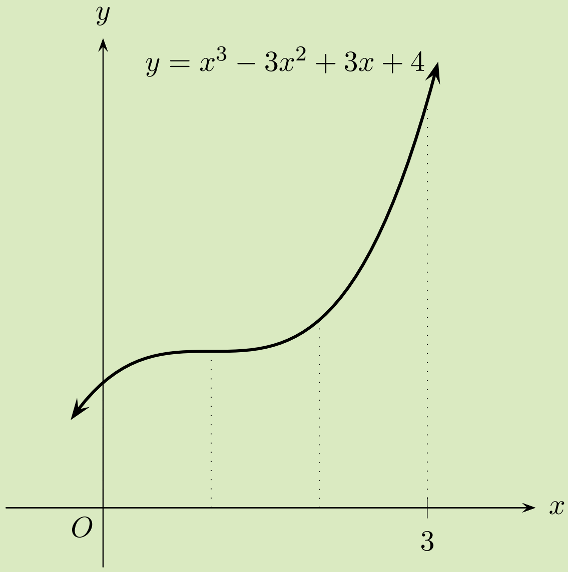
$$\int_0^5 \sqrt{25 - x^2}\, dx$$

iv. Use your answer to part (iii) to explain why the exact value of the integral is $\dfrac{25\pi}{4}$. **1**

v. Use your answers to part (ii) and (iv) to find an approximate value for π. **1**

Example 56

[2023 NBHS Adv Assessment Task 3] The graph below shows the curve $y = x^3 - 3x^2 + 3x + 4$.



(a) Use three applications of the Trapezoidal Rule from $x = 0$ to $x = 3$ to find the approximate area bounded by the curve and the x axis. **3**

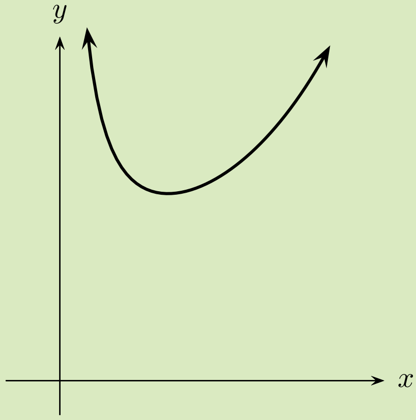
(b) By using the graph provided, give a reason why the Trapezoidal Rule produces an overestimate of the exact area under the curve. **1**

Important note

⚠ How many functions values are needed for 3 applications of the Trapezoidal Rule?

Example 57

[2019 Independent 2U Q16] The diagram shows the graph of the curve $y = \dfrac{e^x}{x}$ for $x > 0$.



i. Find the coordinates of the stationary point on the curve. **2**

ii. Use the Trapezoidal Rule with 4 function values to find as a rational number, an approximation to the area bounded by the curve and the x axis between $x = \ln 2$ and $x = \ln 16$. **3**

Answer: i. $(1, e)$ ii. $\frac{23}{3}$ sq. units

Further exercises

Ⓐ **Ex 4H**
- Q4-15

ⓧ1 **Ex 5A**
- Q8-11

Ex 5H
- Q6-15

6.1.1 Supplementary exercises

1. Draw a sketch of $y = x^2+1$. Find the upper and lower bounds for the integral $\int_0^2 \left(x^2+1\right)\,dx$ using (a) 1 rectangle (b) 2 rectangles (c) 4 rectangles.

2. Repeat Question 1 for $\int_1^3 2^x\,dx$.

3. (a) Use a sketch and one application of the trapezoidal rule to approximate the following integrals to an appropriate number of decimal places.

i. $\int_0^4 x^3\,dx$ ii. $\int_1^3 \log_3 x\,dx$ iii. $\int_2^3 \frac{dx}{x}$

(b) Decide whether each of the approximations in part (a) is an over or under approximation to the exact value of the integral.

(c) Repeat part (a) using two applications of the trapezoidal rule.

4. Use the trapezoidal rule with five function values to approximate:

(a) $\int_0^6 \sqrt{x}\,dx$ (b) $\int_0^1 \frac{dx}{1+x^2}$ (c) $\int_0^{\frac{\pi}{2}} \sin x\,dx$

5. (a) Use the trapezoidal rule with six function values to approximate $\int_0^1 \sqrt{1-x^2}\,dx$, giving your answer correct to 3 decimal places.

(b) Use your answer in part (a) to find an approximation for π.

Extension

6. (a) Complete the following table for the function $y = x^2 - 4x + 3$.

x	0	1	2	3	4
y					

(b) Use the trapezoidal rule with all the values in this table to approximate

i. $\int_0^4 \left(x^2-4x+3\right)\,dx$

ii. the area of the region bounded by the curve $y = x^2 - 4x + 3$, the x axis, and the lines $x = 0$ and $x = 4$.

7. If $f(x) = x + \frac{1}{x}$, use the trapezoidal rule with three function values to approximate $\pi\int_1^2 \left(f(x)\right)^2\,dx$.

8. Use the trapezoidal rule with five function values to approximate the area bounded by the curves $y = x^2$ and $y = x^3$.

Answers to supplementary exercises §6.1.1

1. (a) $LS = 2$, $US = 10$ (b) $LS = 3$, $US = 7$ (c) $LS = \frac{15}{4}$, $US = \frac{23}{4}$ **2.** (a) $LS = 4$, $US = 16$ (b) $LS = 6$, $US = 12$ (c) $LS = 3\left(1+\sqrt{2}\right) \approx 7.24$, $US = 3\left(2+\sqrt{2}\right) \approx 10.24$ **3.** (a) i. 128 ii. 1 iii. 0.42 (b) i. over ii. under iii. over (c) i. 80 ii. 1.13 iii. 0.41 **4.** (a) 9.45 (b) 0.783 (c) 0.987 **5.** (a) 0.759 (b) 3.036 **6.** (a) $3, 0, -1, 0, 3$ (b) i. 2 ii. 4 units^2 **7.** 15.4 **8.** 0.078 units^2

Further exercises (Legacy Textbooks)

Ex 11I Q1-11 **Ex 11J** Q1-12

Section 7

Rates of change

Learning Goal(s)

Knowledge
Identifying when to differentiate and when to integrate when solving rates of change problems

Skills
Solving rates of change problems by integration

Understanding
When to use the definite or indefinite integral to solve rates of change problems

By the end of this section am I able to:

21.18 Integrate functions and find indefinite or definite integrals and apply this technique to solving practical problems

- When given $\frac{dQ}{dt}$, integrate and use initial conditions to find Q.
- When finding the *change* of quantity over the time, use a definite integral.

Example 58

(Pender et al., 2019a, p.390) A tank contains 40 000 L of water. When the draining valve is opened, the volume V in litres of water in the tank decreases at a variable rate given by $\frac{dV}{dt} = -1500 + 30t$, where t is the time in seconds after opening the valve. Once the water stops flowing, the valve shuts off.

(a) When does the water stop flowing?

(b) Give a common-sense reason why the rate $\frac{dV}{dt}$ is negative up to this time.

(c) Integrate to find the volume of water in the tank at time t.

(d) How much water has flowed out of the tank and how much remains?

Example 59

[2006 2U HSC Q9] During a storm, water flows into a 7 000-litre tank at a rate of $\frac{dV}{dt}$ litres per minute, where $\frac{dV}{dt} = 120 + 26t - t^2$ and t is the time in minutes since the storm began.

(i) At what times is the tank filling at twice the initial rate? **2**

(ii) Find the volume of water that has flowed into the tank since the start of the storm as a function of t. **1**

(iii) Initially, the tank contains 1 500 litres of water. When the storm finishes, 30 minutes after it began, the tank is overflowing. **2**

How many litres of water have been lost?

Example 60

[2014 Independent 2U Q16] After a week of rain the local dam starts to fill until, at 10 am Sunday the dam overflows. At this point the height (H) of the river starts to change at the rate of

$$\left(1 - \frac{t}{20}\right) \text{ metres per hour}$$

Initially the height of the river is 5 metres.

i.) Show that the height of the river is given by the formula **1**

$$H = -\frac{t^2}{40} + t + 5$$

ii.) Find the maximum height of the river during this flood. **2**

iii.) A bridge crossing this river will be blocked once the height of the river reaches 12.5 metres. At what times and days will the bridge be blocked and then re-opened? **3**

Answer: i. Show ii. 15 m iii. Blocked: 8 pm Sunday. Opened: 4 pm Monday.

Example 61

[1998 2U HSC Q8] Sand is tipped from a truck onto a pile. The rate, R kg/s, at which the sand is flowing is given by the expression $R = 100t - t^3$, for $0 \leq t \leq T$, where t is the time in seconds after the sand begins to flow.

(i) Find the rate of flow at time $t = 8$. **1**

(ii) What is the largest value of T for which the expression for R is physically reasonable? **1**

(iii) Find the maximum rate of flow of sand. **1**

(iv) When the sand starts to flow, the pile already contains 300 kg of sand. Find an expression for the amount of sand in the pile at time t. **2**

(v) Calculate the total weight of sand that was tipped from the truck in the first 8 seconds. **2**

Example 62

⚠ (Pender, Sadler, Shea, & Ward, 2009, Q20, p.268) James had a full drink bottle containing 500 mL of Coke. He drank from it so that the volume V mL of Coke in the bottle changed at a rate given

$$\frac{dV}{dt} = \left(\frac{2}{5}t - 20\right) \text{ mL/s}$$

(a) Find a formula for V.

(b) Show that it took James 50 seconds to drink the contents of the bottle.

(c) How long, correct to the nearest second, did it take James to drink half the contents of the bottle?

Example 63

[2011 2U HSC Q9] A tap releases liquid A into a tank at the rate of $\left(2 + \dfrac{t^2}{t+1}\right)$ litres per minute, where t is the time in minutes. A second tap releases liquid B into the same tank at the rate of $\left(1 + \dfrac{1}{t+1}\right)$ litres per minute. The taps are opened at the same time and release the liquids into an empty tank.

(i) Show that the rate of flow of liquid A is greater than the rate of flow of liquid B by t litres per minute. **1**

(ii) The taps are closed after 4 minutes. By how many litres is the volume of liquid A greater than the volume of liquid B in the tank when the taps are closed? **2**

Answer: 8 L

Further exercises

Ⓐ **Ex 7E**
- Q3, 4, 5, 7

Ⓧ1 **Ex 9F**
- Q1, 2, 4, 11, 12

Further exercises (Legacy Textbooks)

② **Ex 6F** (Pender et al., 2009)
- Q7-19

Ⓧ1 **Ex 7F** (Pender, Sadler, Shea, & Ward, 2000)
- Q1, 3, 6, 9, 10

Section 8

Motion

8.1 Displacement & velocity as integrals

⚠ Relate content to *Further Differentiation* topic and previous sections.

Laws/Results

The .... - .... function of a particle is the integral of the velocity w.r.t. time, i.e.

$$x = \ldots\ldots\ldots\ldots\ldots$$

(Do not omit constant of integration!)

Laws/Results

The .... between $t = a$ and $t = b$, from a velocity-time equation is

$$\Delta x = \ldots\ldots\ldots\ldots\ldots\ldots$$

(Also see Section 7 on page 65)

Laws/Results

The velocity-time function of a particle is the integral of the acceleration w.r.t. time, i.e.

$$v = \int a\, dt$$

(Do not omit constant of integration!)

Example 64

[2014 St George GHS 2U] A particle moves along a straight horizontal line with acceleration of $(2t - 1)$ ms^{-2}. Initially it is 3 metres to the right of the origin, moving with velocity of -2 ms^{-1}. The position of the particle, relative to the origin after 3 seconds is:

(A) 1.5 m to the right

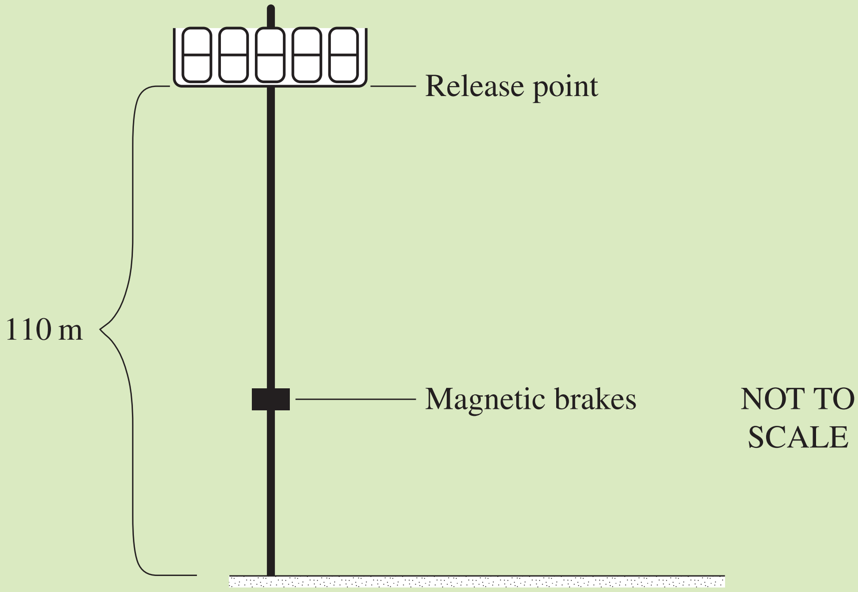
(B) 1.5 m to the left

(C) 11.5 m to the right

(D) 11.5 m to the left

Example 65

[2015 2U HSC Q14] In a theme park ride, a chair is released from a height of 110 metres and falls vertically. Magnetic brakes are applied when the velocity of the chair reaches -37 metres per second.



The height of the chair at time t seconds is x metres. The acceleration of the chair is given by $\ddot{x} = -10$. At the release point, $t = 0$, $x = 110$ and $\dot{x} = 0$.

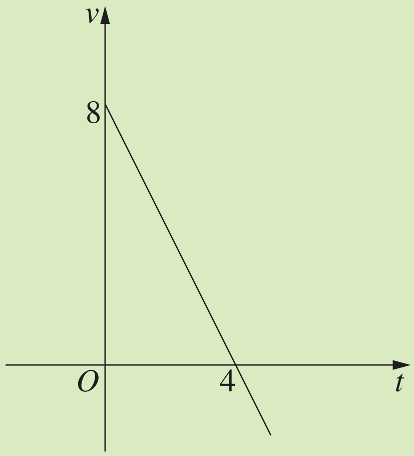
(i) Using calculus, show that $x = -5t^2 + 110$. **2**

(ii) How far has the chair fallen when the magnetic brakes are applied? **2**

Answer: (i) Show (ii) 68.45 m

Example 66

[2020 Adv HSC Sample Q33] (2 marks) A particle is moving along the x axis. The graph shows its velocity v metres per second at time t seconds.



When $t = 0$ the displacement x is equal to 2 metres.

On the axes below draw a graph that shows the particle's displacement, x metres from the origin, at a time t seconds between $t = 0$ and $t = 4$. Label the coordinates of the endpoints of your graph.

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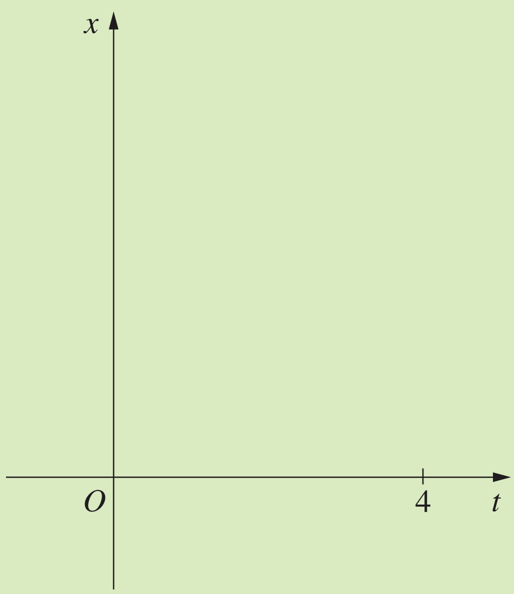


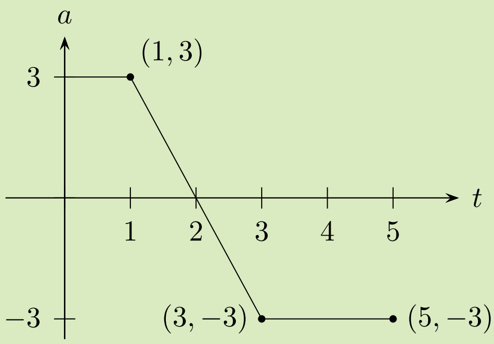
8.2 Distance travelled

Laws/Results

The distance travelled can be found by finding the .... between the graph and .... axis.

Example 67

[2004 2U HSC] A particle moves along the x-axis. Initially it is at rest at the origin. The graph shows the acceleration, a, of the particle as a function of time t for $0 \leq t \leq 5$.

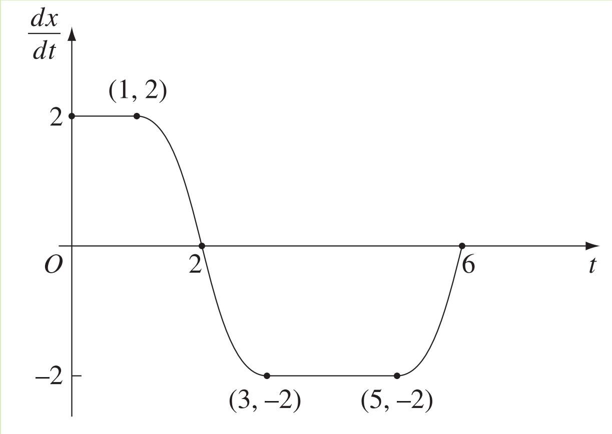


(i) Write down the time at which the velocity of the particle is a maximum. **1**

(ii) At what time during the interval $0 \leq t \leq 5$ is the particle furthest from the origin? Give brief reasons for your answer. **2**

Example 68

[2005 2U HSC Q7] The graph shows the velocity $\frac{dx}{dt}$, of a particle as a function of time. Initially, the particle is at the origin.



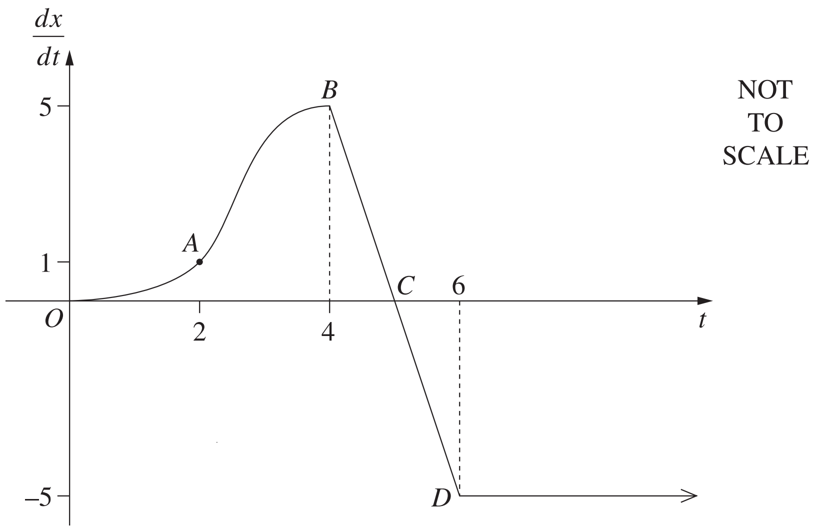
(i) At what time is the displacement, x, from the origin a maximum? **1**

(ii) At what time does the particle return to the origin? Justify your answer. **2**

(iii) Draw a sketch of the acceleration, $\frac{d^2x}{dt^2}$ as a function of time for $0 \leq t \leq 6$. **2**

Additional questions

1. **[2007 2U HSC Q10]** An object is moving on the x axis. The graph show the velocity $\dfrac{dx}{dt}$, of the object, as a function of time t. The coordinates of the points shown on the graph are $A(2,1)$, $B(4,5)$, $C(5,0)$ and $D(6,-5)$. The velocity is constant for $t \geq 6$.



(i) Using ~~Simpson's Rule~~ the Trapezoidal Rule, estimate the distance travelled between $t = 0$ and $t = 4$. **2**

(ii) The object is initially at the origin. During which time(s) is the displacement of the object decreasing? **1**

(iii) Estimate the time at which the object returns to the origin. Justify your answer. **2**

(iv) Sketch the displacement, x, as a function of time. **2**

Further exercises

Ⓐ **Ex 7C**

- Q1-7
- Q8(a), (b), (g)
- Q9(a), (b), (g), (h)
- Q9(a), (b), (g)
- Q10-14

(x1) **Ex 9C**

- Q1-3
- Q4(a)iv, (b)i, iv
- Q5-10, 15, Q18

NSW Education Standards Authority

2020 HIGHER SCHOOL CERTIFICATE EXAMINATION

Mathematics Advanced
Mathematics Extension 1
Mathematics Extension 2

REFERENCE SHEET

Measurement

Length

$l = \frac{\theta}{360} \times 2\pi r$

Area

$A = \frac{\theta}{360} \times \pi r^2$

$A = \frac{h}{2}(a + b)$

Surface area

$A = 2\pi r^2 + 2\pi rh$

$A = 4\pi r^2$

Volume

$V = \frac{1}{3}Ah$

$V = \frac{4}{3}\pi r^3$

Functions

$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

For $ax^3 + bx^2 + cx + d = 0$:

$\alpha + \beta + \gamma = -\frac{b}{a}$

$\alpha\beta + \alpha\gamma + \beta\gamma = \frac{c}{a}$

and $\alpha\beta\gamma = -\frac{d}{a}$

Relations

$(x - h)^2 + (y - k)^2 = r^2$

Financial Mathematics

$A = P(1 + r)^n$

Sequences and series

$T_n = a + (n - 1)d$

$S_n = \frac{n}{2}[2a + (n - 1)d] = \frac{n}{2}(a + l)$

$T_n = ar^{n-1}$

$S_n = \frac{a(1 - r^n)}{1 - r} = \frac{a(r^n - 1)}{r - 1}, \ r \neq 1$

$S = \frac{a}{1 - r}, \ |r| < 1$

Logarithmic and Exponential Functions

$\log_a a^x = x = a^{\log_a x}$

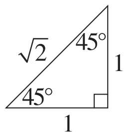
$\log_a x = \frac{\log_b x}{\log_b a}$

$a^x = e^{x \ln a}$

Trigonometric Functions

$\sin A = \frac{\text{opp}}{\text{hyp}}, \quad \cos A = \frac{\text{adj}}{\text{hyp}}, \quad \tan A = \frac{\text{opp}}{\text{adj}}$

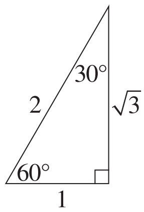
$A = \frac{1}{2}ab\sin C$



$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$

$c^2 = a^2 + b^2 - 2ab\cos C$

$\cos C = \frac{a^2 + b^2 - c^2}{2ab}$



$l = r\theta$

$A = \frac{1}{2}r^2\theta$

Trigonometric identities

$\sec A = \frac{1}{\cos A}, \ \cos A \neq 0$

$\operatorname{cosec} A = \frac{1}{\sin A}, \ \sin A \neq 0$

$\cot A = \frac{\cos A}{\sin A}, \ \sin A \neq 0$

$\cos^2 x + \sin^2 x = 1$

Compound angles

$\sin(A + B) = \sin A\cos B + \cos A\sin B$

$\cos(A + B) = \cos A\cos B - \sin A\sin B$

$\tan(A + B) = \frac{\tan A + \tan B}{1 - \tan A\tan B}$

If $t = \tan\frac{A}{2}$ then $\sin A = \frac{2t}{1+t^2}$

$\cos A = \frac{1-t^2}{1+t^2}$

$\tan A = \frac{2t}{1-t^2}$

$\cos A\cos B = \frac{1}{2}\left[\cos(A - B) + \cos(A + B)\right]$

$\sin A\sin B = \frac{1}{2}\left[\cos(A - B) - \cos(A + B)\right]$

$\sin A\cos B = \frac{1}{2}\left[\sin(A + B) + \sin(A - B)\right]$

$\cos A\sin B = \frac{1}{2}\left[\sin(A + B) - \sin(A - B)\right]$

$\sin^2 nx = \frac{1}{2}(1 - \cos 2nx)$

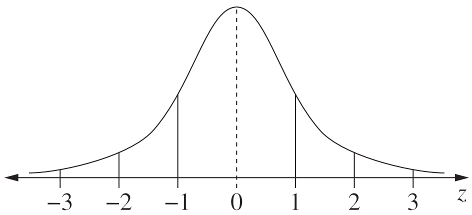
$\cos^2 nx = \frac{1}{2}(1 + \cos 2nx)$

Statistical Analysis

$z = \frac{x - \mu}{\sigma}$

An outlier is a score
less than $Q_1 - 1.5 \times IQR$
or
more than $Q_3 + 1.5 \times IQR$

Normal distribution



- approximately 68% of scores have z-scores between –1 and 1
- approximately 95% of scores have z-scores between –2 and 2
- approximately 99.7% of scores have z-scores between –3 and 3

$E(X) = \mu$

$\operatorname{Var}(X) = E\left[(X - \mu)^2\right] = E\left(X^2\right) - \mu^2$

Probability

$P(A \cap B) = P(A)P(B)$

$P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$P(A|B) = \frac{P(A \cap B)}{P(B)}, \ P(B) \neq 0$

Continuous random variables

$P(X \le x) = \int_a^x f(x)\,dx$

$P(a < X < b) = \int_a^b f(x)\,dx$

Binomial distribution

$P(X = r) = {}^nC_r p^r(1 - p)^{n-r}$

$X \sim \operatorname{Bin}(n, p)$

$\Rightarrow \ P(X = x)$

$= \binom{n}{x} p^x(1 - p)^{n-x}, \ x = 0, 1, \ldots, n$

$E(X) = np$

$\operatorname{Var}(X) = np(1 - p)$

Differential Calculus

Function	Derivative
$y = f(x)^n$	$\frac{dy}{dx} = nf'(x)\left[f(x)\right]^{n-1}$
$y = uv$	$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$
$y = g(u)$ where $u = f(x)$	$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$
$y = \frac{u}{v}$	$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$
$y = \sin f(x)$	$\frac{dy}{dx} = f'(x)\cos f(x)$
$y = \cos f(x)$	$\frac{dy}{dx} = -f'(x)\sin f(x)$
$y = \tan f(x)$	$\frac{dy}{dx} = f'(x)\sec^2 f(x)$
$y = e^{f(x)}$	$\frac{dy}{dx} = f'(x)e^{f(x)}$
$y = \ln f(x)$	$\frac{dy}{dx} = \frac{f'(x)}{f(x)}$
$y = a^{f(x)}$	$\frac{dy}{dx} = (\ln a)f'(x)a^{f(x)}$
$y = \log_a f(x)$	$\frac{dy}{dx} = \frac{f'(x)}{(\ln a)f(x)}$
$y = \sin^{-1} f(x)$	$\frac{dy}{dx} = \frac{f'(x)}{\sqrt{1-\left[f(x)\right]^2}}$
$y = \cos^{-1} f(x)$	$\frac{dy}{dx} = -\frac{f'(x)}{\sqrt{1-\left[f(x)\right]^2}}$
$y = \tan^{-1} f(x)$	$\frac{dy}{dx} = \frac{f'(x)}{1+\left[f(x)\right]^2}$

Integral Calculus

$$\int f'(x)\left[f(x)\right]^n dx = \frac{1}{n+1}\left[f(x)\right]^{n+1} + c$$

where $n \neq -1$

$$\int f'(x)\sin f(x)\,dx = -\cos f(x) + c$$

$$\int f'(x)\cos f(x)\,dx = \sin f(x) + c$$

$$\int f'(x)\sec^2 f(x)\,dx = \tan f(x) + c$$

$$\int f'(x)e^{f(x)}dx = e^{f(x)} + c$$

$$\int \frac{f'(x)}{f(x)}dx = \ln\left|f(x)\right| + c$$

$$\int f'(x)a^{f(x)}dx = \frac{a^{f(x)}}{\ln a} + c$$

$$\int \frac{f'(x)}{\sqrt{a^2 - \left[f(x)\right]^2}}dx = \sin^{-1}\frac{f(x)}{a} + c$$

$$\int \frac{f'(x)}{a^2 + \left[f(x)\right]^2}dx = \frac{1}{a}\tan^{-1}\frac{f(x)}{a} + c$$

$$\int u\frac{dv}{dx}dx = uv - \int v\frac{du}{dx}dx$$

$$\int_a^b f(x)\,dx$$

$$\approx \frac{b-a}{2n}\left\{f(a) + f(b) + 2\left[f(x_1) + \cdots + f(x_{n-1})\right]\right\}$$

where $a = x_0$ and $b = x_n$

Combinatorics

$$^nP_r = \frac{n!}{(n-r)!}$$

$$\binom{n}{r} = {}^nC_r = \frac{n!}{r!(n-r)!}$$

$$(x+a)^n = x^n + \binom{n}{1}x^{n-1}a + \cdots + \binom{n}{r}x^{n-r}a^r + \cdots + a^n$$

Vectors

$$\left|\underset{\sim}{u}\right| = \left|x\underset{\sim}{i} + y\underset{\sim}{j}\right| = \sqrt{x^2 + y^2}$$

$$\underset{\sim}{u} \cdot \underset{\sim}{v} = \left|\underset{\sim}{u}\right|\left|\underset{\sim}{v}\right|\cos\theta = x_1x_2 + y_1y_2,$$

where $\underset{\sim}{u} = x_1\underset{\sim}{i} + y_1\underset{\sim}{j}$

and $\underset{\sim}{v} = x_2\underset{\sim}{i} + y_2\underset{\sim}{j}$

$$\underset{\sim}{r} = \underset{\sim}{a} + \lambda\underset{\sim}{b}$$

Complex Numbers

$$z = a + ib = r(\cos\theta + i\sin\theta)$$
$$= re^{i\theta}$$

$$\left[r(\cos\theta + i\sin\theta)\right]^n = r^n(\cos n\theta + i\sin n\theta)$$
$$= r^ne^{in\theta}$$

Mechanics

$$\frac{d^2x}{dt^2} = \frac{dv}{dt} = v\frac{dv}{dx} = \frac{d}{dx}\left(\frac{1}{2}v^2\right)$$

$$x = a\cos(nt + \alpha) + c$$

$$x = a\sin(nt + \alpha) + c$$

$$\ddot{x} = -n^2(x - c)$$

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